Fundamental Dominations in Graphs

Arash Behzad
University of California, LosAngeles
abehzad@ee.ucla.edu

Mehdi Behzad Shahid Beheshti University, Iran mbehzad@sharif.edu

Cheryl E. Praeger
University of Western Australia, Australia
praeger@maths.uwa.edu.au

Abstract

Nine variations of the concept of domination in a simple graph are identified as fundamental domination concepts, and a unified approach is introduced for studying them. For each variation, the minimum cardinality of a subset of dominating elements is the corresponding fundamental domination number. It is observed that, for each nontrivial connected graph, at most five of these nine numbers can be different, and inequalities between these five numbers are given. Finally, these fundamental dominations are interpreted in terms of the total graph of the given graph, a concept introduced by the second author in 1965. It is argued that the very first domination concept, defined by O. Ore in 1962 and under a different name by C. Berge in 1958, deserves

to be called the most fundamental of graph dominations.

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1 Introduction and Preliminaries

The literature contains extensive studies of many variations of the concept of

domination in a simple graph. The following points are some loose extracts

from the Preface, Chapter 12, and the Appendix of the reference text [9],

written by well-known authorities: T.W. Haynes, S.T. Hedetniemi, and P.J.

Slater.

• One of the authors' objectives is "to consolidate and organize much of

the material in the more than 1200 papers already published on domination

in graphs."

• "It is well known and generally accepted that the problem of determining

the domination number of an arbitrary graph is a difficult one. ... Because of

this, researchers have turned their attention to the study of classes of graphs

for which the domination problem can be solved in polynomial time".

• "The following pages contain a fairly comprehensive census of more

than 75 models of dominating and related types of sets in graphs which have

appeared in the research literature over the past 20 years."

Without a doubt, the literature on this subject is growing rapidly, and a

considerable amount of work has been dedicated to find different bounds for

the domination numbers of graphs [3, 8, 10, 11]. The terms "dominating set",

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and "domination number" of a graph G = (V, E) were first defined by O. Ore in 1962, see [12]. A subset $A \subseteq V$ is a dominating set for G if each element of V is either in A, or is adjacent to an element of A. The domination number $\gamma(G)$, which is the most commonly used domination number, is the minimum cardinality among all dominating sets of G. We will interpret $\gamma(G)$ as the minimum cardinality among all subsets $A \subseteq V$ dominating the set $A \subseteq V$. Such a subset A and the set $A \subseteq V$ are called a dominating set and the dominated set for this vertex-vertex domination variation, respectively. The parameter $\gamma(G)$ will be referred to henceforth as the vertex-vertex domination number of G.

Later, a few researchers defined, sometimes redefined, and studied other domination variations: vertex-edge, edge-vertex, etc., and the cardinalities of their largest or smallest dominating sets [1,10].

However, from practical point of view, it was necessary to define other types of dominations. Most of these new variations required the dominating set to have additional properties such as: being as independent set, inducing a connected subgraph, or inducing a clique. These properties were reflected in their names as an adjective: independent domination, connected domination, and clique domination, respectively.

In this paper, among over 75 models of domination and corresponding subsets in graphs, we choose nine variations, which are the core, and call them fundamental dominations. For this purpose, we consider simple nontrivial connected graphs G = (V, E). Our reason for restricting attention to such graphs are two-fold. First, two of the fundamental domination numbers are not defined for graphs with isolated vertices. Second, if G is disconnected and has no isolated vertices, then the value of each of the fundamental domination numbers for G is equal to the sum of the values of the same number for each

of the connected components of G.

We introduce a unified approach to studying these fundamental dominations based on the fact that, in each domination variation, two sets are used - the set consisting of the dominating elements, and the set consisting of the elements that need to be dominated. For each fundamental domination the minimum cardinality among all dominating sets is the corresponding fundamental domination number. We observe, in Theorem 1, that for each nontrivial connected graph at most five of these nine numbers can be different. Inequalities concerning each pair of these five numbers are considered in Theorems 2 and 3.

Finally, we show how these fundamental dominations may be interpreted in terms of the total graph T(G) of G, defined by the second author in 1965. We argue that the very first domination concept, defined by O. Ore in 1962 and under a different name by C. Berge in 1958, deserves to be called the most fundamental of graph dominations.

1.1 The fundamental domination numbers

By an element of a graph G = (V, E) we mean a member of the set $V \cup E$. Two different elements of G are said to be associated if they are adjacent or incident in G. For $U, W \in \{V, E, V \cup E\}$, a subset $A \subseteq U$ dominates W if each element of $W \setminus (A \cap W)$ is associated with an element of A. The minimum cardinality of such subsets A is denoted by $\gamma_{U,W}(G)$. For historical reasons, we replace $\gamma_{V,V}(G)$, $\gamma_{E,E}(G)$, and $\gamma_{V \cup E,V \cup E}(G)$ by $\gamma(G)$, $\gamma'(G)$, and $\gamma''(G)$, respectively. We call the following nine parameters the fundamental

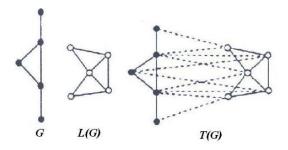


Figure 1: A graph G, its line graph L(G) and total graph T(G)

domination numbers of G:

$$\begin{split} \gamma_{\scriptscriptstyle V,V}(G) &= \gamma(G); \quad \gamma_{\scriptscriptstyle V,E}(G); \quad \gamma_{\scriptscriptstyle V,V\cup E}(G); \\ \gamma_{\scriptscriptstyle E,V}(G); \quad \gamma_{\scriptscriptstyle E,E}(G) &= \gamma^{'}(G); \quad \gamma_{\scriptscriptstyle E,V\cup E}(G); \\ \gamma_{\scriptscriptstyle V\cup E,V}(G); \gamma_{\scriptscriptstyle V\cup E,E}(G); \quad \gamma_{\scriptscriptstyle V\cup E,V\cup E}(G) &= \gamma^{''}(G). \end{split}$$

In each case, a dominating subset whose cardinality equals the fundamental domination number is called a fundamental dominating subset of G.

Various equalities and inequalities related to these fundamental numbers are summarized in Theorems 1, 2, and 3 of Section 2. See Figure 2 which actually reflects the statements of these three theorems.

The line graph L(G) of a nonempty graph G = (V, E) is the graph whose vertex set is in one-to-one correspondence with the elements of the set E such that two vertices of L(G) are adjacent if and only if they correspond to two adjacent edge of G. The total graph T(G) of G is the graph whose vertex set is in one-to-one correspondence with the set $V \cup E$ of elements of G such that two vertices of T(G) are adjacent if and only if they correspond to two adjacent or incident elements of G, see [4,6]. Figure 1 shows a graph G, its line graph L(G) and its total graph T(G). It is clear that G and L(G) are disjoint induced subgraphs of T(G).

In Section 3, we elaborate on the fact that each graph domination variation can be presented in the context of a special subset A of V(T(G)) dominating an appropriate subset B of V(T(G)). This observation suggests

that the phrase "the most fundamental of graph dominations" might appropriately be attached to the vertex-vertex domination related to γ .

Notions and notations not defined here can be found in texts such as M. Behzad, et al [6], and D. B. West [14].

2 General Equalities and Inequalities

Each of the nine fundamental domination numbers has its own applications, and must be considered separately. However, for each graph under consideration the following equalities hold.

Theorem 1 For each nontrivial connected graph G we have:

$$(1) \ \gamma'(G) = \gamma(L(G)), \qquad (2) \ \gamma''(G) = \gamma(T(G)), \qquad (3) \ \gamma_{V \cup E, V}(G) = \gamma(G), \\ (4) \ \gamma_{V \cup E, E}(G) = \gamma'(G), \qquad (5) \ \gamma_{E, V}(G) = \gamma_{E, V \cup E}(G), \qquad (6) \ \gamma_{V, E}(G) = \gamma_{V, V \cup E}(G).$$

Proof. The equalities (1), and (2) follow from the definitions of the line graph and the total graph of G, respectively. To prove the last four equalities, first we observe that, by definition, the parameter on the left of each equation in less than or equal to the parameter on the right.

For example, in (3) we show that $\gamma_{V \cup E, V}(G) \leq \gamma(G) = \gamma_{V,V}(G)$. Since each subset of V dominating V is a subset of $V \cup E$ that dominates V, the inequality $\gamma_{V \cup E, V}(G) \leq \gamma_{V,V}(G) = \gamma(G)$ follows. The same is true for the inequality $\gamma_{V \cup E, E}(G) \leq \gamma'(G) = \gamma_{E,E}(G)$.

Next, we prove that $\gamma_{E,V}(G) \leq \gamma_{E,V \cup E}(G)$. Subsets such as A of the set E that dominates V do not necessarily dominates $V \cup E$. Ordinarily, subsets of bigger size are needed to dominate both V and E. Thus by definitions involved $\gamma_{E,V}(G) \leq \gamma_{E,V \cup E}(G)$. The same argument shows that $\gamma_{V,E}(G) \leq \gamma_{E,V \cup E}(G)$.

 $\gamma_{V,V\cup E}(G)$.

To complete the proofs of the equalities (3)-(6) it suffices to prove the validity of each converse inequality. As an example, for (3) we show that $\gamma(G) \leq \gamma_{V \cup E, V}(G)$. Let A be a subset of $V \cup E$ which dominates V such that $|A| = \gamma_{V \cup E, V}(G)$. If $A \cap E = \phi$, then $A \subseteq V$, and $\gamma(G) \leq \gamma_{V \cup E, V}(G)$. Suppose $A \cap E \neq \phi$. For each $e = uv \in A \cap E$ eliminate e from A and, if necessary, add to the remaining subset one of u or v to produce a set $A' \subseteq V$ dominating V. Thus $\gamma(G) \leq |A'| \leq |A| = \gamma_{V \cup E, V}(G)$.

Next, we prove that $\gamma'(G) \leq \gamma_{V \cup E,E}(G)$. Let $A \subseteq V \cup E$ which dominates E such that $|A| = \gamma_{V \cup E,V}(G)$. If $A \cap V = \phi$, then $A \subseteq E$, and $\gamma'(G) \leq \gamma_{V \cup E,E}(G)$. Otherwise, $A \cap V \neq \phi$. For each $v \in A \cap V$, eliminate v from A, and if necessary, add to the remaining subset an edge incident with v to produce a set $A' \subseteq E$ dominating E. Such an edge exists, since G has no isolated vertices. Thus, as before, $\gamma'(G) \leq |A'| \leq |A| = \gamma_{V \cup E,V}(G)$, and (4) is established.

To Prove $\gamma_{E,V\cup E}(G) \leq \gamma_{E,V}(G)$, let $A \subseteq E$ dominates V, such that $|A| = \gamma_{E,V}(G)$. We claim that A dominates E as well. Let $e = uv \in E \backslash A$. Since A dominates V, there exists an edge $e' \in A$ such that e' and u are incident, that is to say $u \in e'$. Then e and e' are adjacent. Thus A dominates $V \cup E$. Hence $\gamma_{E,V\cup E}(G) \leq \gamma_{E,V}(G)$.

Finally, in a similar manner, we prove that $\gamma_{V,V\cup E}(G) \leq \gamma_{V,E}(G)$. Assume that $A \subseteq V$ dominates E, and that $|A| = \gamma_{V,E}(G)$. We show that A dominates V, too. Let $v \in V \setminus A$. Since G is connected and nontrivial, there exists at least one edge e = uv incident with v, and since A dominates E, one of the vertices u, v must lie in A. Since $v \notin A, u$ must be in A. Hence A dominates $V \cup E$. Therefore, $\gamma_{V,V \cup E}(G) \leq |A| = \gamma_{V,E}(G)$.

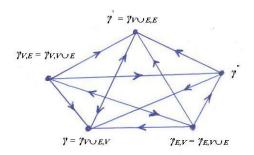


Figure 2: Digraph D to be used in Theorems 2 and 3

Based on Theorem 1, the values of the nine fundamental numbers are essentially reduced to five:

$$\gamma = \gamma_{V \cup E, V}, \gamma_{V, E} = \gamma_{V, V \cup E}, \gamma_{E, V} = \gamma_{E, V \cup E}, \gamma' = \gamma_{V \cup E, E}, \text{ and } \gamma''.$$

Next, we introduce a digraph D with node set $N=\{\gamma,\gamma^{'},\gamma^{''},\gamma_{\scriptscriptstyle V,E},\gamma_{\scriptscriptstyle E,V}\},$ and arc set

$$A = \{ (\gamma_{V,E}, \gamma^{'}), (\gamma_{V,E}, \gamma^{''}), (\gamma_{V,E}, \gamma), (\gamma_{V,E}, \gamma_{E,V}), (\gamma_{E,V}, \gamma_{V,E}), (\gamma^{'}, \gamma), (\gamma, \gamma^{'}), (\gamma_{E,V}, \gamma^{'}), (\gamma^{''}, \gamma^{'}), (\gamma_{E,V}, \gamma^{''}), (\gamma^{''}, \gamma), (\gamma_{E,V}, \gamma) \}.$$

See Figure 2. In this figure the two arcs (γ, γ') , and (γ', γ) are represented by the straight line joining the two nodes γ and γ' along with two arrows in opposite directions. This situation will be denoted by $\gamma \leftrightarrow \gamma'$, or by $\gamma' \leftrightarrow \gamma$. The two arcs $(\gamma_{V,E}, \gamma_{E,V})$ and $(\gamma_{E,V}, \gamma_{V,E})$ are represented in the same way. However, for each of the remaining eight arcs (x, y) in A such that $(y, x) \notin A$, the arc (x, y) is represented by a straight line joining the two nodes x and yalong with one arrow from x to y. This situation will be denoted by $x \to y$.

Theorem 2 Let x and y be two different nodes of the digraph D of Figure 2. If (x, y) is an arc of D and (y, x) is not an arc of D, then for every nontrivial connected graph G, the inequality $x(G) \ge y(G)$ holds.

Proof. The equalities stated in Theorem 1 are depicted in Figure 2. In the light of these equalities, it suffices to show that the following eight inequalities

are valid:

$$(1) \quad \gamma_{V,E}(G) \ge \gamma_{V \cup E,E}(G), \qquad (2) \quad \gamma_{V,V \cup E}(G) \ge \gamma''(G),$$

(3)
$$\gamma_{E,V}(G) \ge \gamma_{V \cup E,V}(G)$$
, (4) $\gamma_{E,V \cup E}(G) \ge \gamma''(G)$,
(5) $\gamma_{V,V \cup E}(G) \ge \gamma(G)$, (6) $\gamma''(G) \ge \gamma_{V \cup E,V}(G)$,
(7) $\gamma_{E,V \cup E}(G) \ge \gamma'(G)$, (8) $\gamma''(G) \ge \gamma_{V \cup E,E}(G)$.

(5)
$$\gamma_{V,V \cup E}(G) \ge \gamma(G)$$
, (6) $\gamma''(G) \ge \gamma_{V \cup E,V}(G)$,

(7)
$$\gamma_{E,V \cup E}(G) \ge \gamma'(G)$$
, (8) $\gamma''(G) \ge \gamma_{V \cup E,E}(G)$

Proofs of the validity of each of the first four inequalities are similar. "Having more freedom to choose the required dominating sets" is the key phrase. Note that in both sides of each of these inequalities the same set needs to be dominated; however, each dominating set of the parameter on the left is necessarily a dominating set of the parameter on the right too. As an example, (2) holds, since $\gamma'' = \gamma_{V \cup E, V \cup E}, V \cup E = V \cup E$, and $V \subseteq V \cup E$.

Similar arguments prove the last four inequalities. For each inequality, the dominating sets of the two parameters which appear on either side are subsets of the same set; however, for the parameter on the right the set which needs to be dominated is a subset of the same set for the parameter on the left. As an example, (7) holds since $\gamma' = \gamma_{E,E}, E = E$, and $E \subseteq V \cup E$.

In the sequel, we will refer to the following special classes of graphs, along with some of their specified domination numbers. We will then use these considerations in Theorem 3 to explain the significance of the two "double arcs" $x \leftrightarrow y$ in Figure 2.

Example 1. For a complete graph, K_n , of order $n, n \geq 2$, one can readily see that:

$$\gamma(K_n) = 1; \quad \gamma'(K_n) = \left\lfloor \frac{n}{2} \right\rfloor;$$

$$\gamma_{E,V}(K_n) = \left\lceil \frac{n}{2} \right\rceil; \quad \gamma_{V,E}(K_n) = n - 1.$$

Example 2. It is easy to observe that for the complete bipartite graph $K_{1,n}, n \in \mathbb{N}$, we have

$$\gamma_{VE}(K_{1,n}) = 1; \quad \gamma_{EV}(K_{1,n}) = n.$$

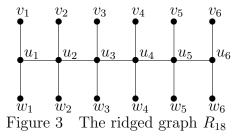
Example 3. Next we introduce a new class of graphs, R_{3n} , $n \in \mathbb{N}$, called *ridged graphs*. Let:

$$V(R_{3n}) = \{u_i, v_i, w_i | 1 \le i \le n\}, \text{ and }$$

$$E(R_{3n}) = E_1 \cup E_2 \cup E_3$$
, where

$$E_1 = \{u_i u_{i+1} | 1 \le i \le n-1\}, \ E_2 = \{u_i v_i | 1 \le i \le n\}, \ E_3 = \{u_i w_i | 1 \le i \le n\}.$$

The graph R_{18} is shown in Figure 3.



By an easy induction on n one can prove that:

$$\gamma(R_{3n}) = n; \text{ and } \gamma'(R_{3n}) = \left\lceil \frac{n}{2} \right\rceil.$$

Theorem 3 If $x \leftrightarrow y$ is a double arc of the digraph D of Figure 2, then for each positive integer r there exist graphs G and H such that x(G) - y(G) > r, and y(H) - x(H) > r.

Proof. The digraph D contains two double arcs of the form $x \leftrightarrow y$: namely $\gamma' \leftrightarrow \gamma$, and $\gamma_{V,E} \leftrightarrow \gamma_{E,V}$. Hence, for a given r we must provide four graphs G_1, G_2, G_3 , and G_4 such that

$$\gamma'(G_1) - \gamma(G_1) > r;$$

$$\gamma(G_2) - \gamma'(G_2) > r;$$

$$\gamma_{V,E}(G_3) - \gamma_{E,V}(G_3) > r;$$

$$\gamma_{E,V}(G_4) - \gamma_{V,E}(G_4) > r.$$

To present G_1 , let n = 2r + 4, and $G_1 = K_n$. As specified in Example 1, we have: $\gamma'(K_n) - \gamma(K_n) = \left|\frac{2r+4}{2}\right| - 1 = r+1 > r$.

For G_2 , let n=4r, and $G_2=R_{3n}$. Then Example 3 indicates that: $\gamma(R_{3n})-\gamma'(R_{3n})=n-\left\lceil\frac{n}{2}\right\rceil=2r>r$.

To provide G_3 , as for G_1 , let n=2r+4, and $G_3=K_n$. Then, by Example 1: $\gamma_{V,E}(K_n)-\gamma_{E,V}(K_n)=n-1-\left\lceil\frac{n}{2}\right\rceil=r+1>r$.

Finally let n=r+2, and $G_4=K_{1,n}$. Then, Example 2 implies that: $\gamma_{E,V}(K_{1,n})-\gamma_{V,E}(K_{1,n})=n-1=r+1>r$

3 The Most Fundamental of Graph Dominations

The total graph T(G) of a nonempty graph G = (V, E) was defined in Section 1. It has two vertex disjoint subgraphs G^* and L^* such that G^* and L^* are, respectively, isomorphic to G and to the line graph L(G). See Figure 1, and for a characterization of total graphs see [5].

Each graph domination concept can be presented in the context of a special subset A of the vertex set $V \cup E$ of T(G) dominating an appropriate

subset B of $V \cup E$. For the nine Fundamental domination variations these special subsets A and B are specified below:

$$\begin{split} A &\subseteq V(G^*) \subset V(T(G)) \quad \text{for} \quad \gamma(G), \gamma_{V,E}(G), \quad \text{and} \quad \gamma_{V,V \cup E}(G); \\ A &\subseteq V(L^*) \subset V(T(G)) \quad \text{for} \quad \gamma_{E,V}(G), \gamma^{'}(G), \quad \text{and} \quad \gamma_{E,V \cup E}(G); \\ A &\subseteq V(T(G)) \qquad \quad \text{for} \quad \gamma_{V \cup E,V}(G), \gamma_{V \cup E,E}(G), \quad \text{and} \quad \gamma^{''}(G); \\ B &= V(G^*) \subset V(T(G)) \quad \text{for} \quad \gamma(G), \gamma_{E,V}(G), \quad \text{and} \quad \gamma_{V \cup E,V}(G); \\ B &= V(L^*) \subset V(T(G)) \quad \text{for} \quad \gamma_{V,E}(G), \gamma^{'}(G), \quad \text{and} \quad \gamma_{V \cup E,E}(G); \\ B &= V(T(G)) \qquad \qquad \text{for} \quad \gamma_{V,V \cup E}(G), \gamma_{E,V \cup E}(G), \quad \text{and} \quad \gamma^{''}(G). \end{split}$$

Hence, for any fundamental domination number of G one can simply minimize the cardinalities of special subsets of V(T(G)) that dominate an appropriate subset of V(T(G)). Thus a slight modification of the original concept defined in 1962 by Ore [12], and in 1958 under a different name by Berge [7], deserves to be referred to as the most fundamental of graph dominations.

Concluding Remarks

1. In this paper we have attempted to categorize domination concepts into nine categories. For a nontrivial connected graph G = (V, E), if we use V for the set of vertices, and E for the set of edges of G, then the nine categories might be referred to as dominations in: V - V, V - E, $V - V \cup E$, E - V, E - E, and E - E, and E - E contexts. The majority of the concepts and results in this vast area are related to E - E dominations; see the book by S. T. Hedetniemi and R. C. Laskar [11]. Many of the concepts and results can easily be transformed into other categories.

As an example, we refer to the k-domination defined in the V-V context as follows. For a positive integer k, a subset $S\subseteq V$ is a k-dominating set if for each $u\in V-S$, $|N(u)\cap S|\geq k$, where N(u) denotes the set of vertices of G which are adjacent to u. The k-domination number $\gamma_k(G)$ was considered by E. J. Cockaye, B. Gamble, and B. Shepherd in [8]. Considering this parameter in other contexts and obtaining their values seems to be of interest.

2. Theorems 1, 2, and 3 provide relationships among the nine fundamental domination numbers of a nontrivial connected graph G. One can often combine some of the given bounds with some of the existing results to produce new bounds which might, or might not be sharp. As an example we consider $\gamma''(G) = \gamma(T(G))$, and the following result of C. Payan [13]: if G has order n, size m, and minimum degree δ , then $\gamma(G) \leq (n+2-\delta)/2$. Since the graph T(G) has order n+m, and its minimum degree is 2δ , we have $\gamma''(G) \leq 1 - \delta + (n+m)/2$. Hence Theorems 1, and 2 imply that for a nontrivial connected graph G we have:

$$\gamma^{'}(G)=\gamma_{{\scriptscriptstyle V\cup E,E}}(G)\leq \gamma^{''}(G)=\gamma(T(G))\leq 1-\delta+(n+m)/2.$$

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